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APPLICATION OF THE METHOD OF INFLUENCE FUNCTIONS IN PROBLEMS OF THE THEORY OF CRACKS FOR ANISOTROPIC PLATES

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Application of analytical methods to estimate the strength of composite materials with cracks and fine inclusions is difficult due to the lack of information concerning the distribution of stresses in a neighborhood of cracktips and inclusions of complex configuration in anisotropic materials. A discussion of this problem and a survey of papers in this direction (mainly for rectilinear cracks and inclusions) can be found, for example, in [1-5].

In what follows, based on the method of influence functions, we present a solution of fundamental problems of planar elasticity theory for anisotropic bodies weakened by curvilinear cuts. Integral representations are constructed, which make it possible to formulate uniformly a solving system of singular integral equations (SIE) for the first, second, and mixed problems of elasticity theory. The effectiveness of the integral representations constructed and of the algorithms presented for numerically solving the resulting SIE is demonstrated by solving a number of problems of crack theory for anisotropic plates.

1. We consider an infinite rectilinear-anisotropic plate weakened by a system of smooth curvilinear nonintersecting cuts $L_j = (a_j, b_j)$, $j = \overline{1, n}$ (Fig. 1). We denote the angle between Ox and the normal n to the left edge of the cut of point $t \in L = \bigcup_{j=1}^{n} L_j$ by $\varphi(t)$. We determine the stress-deformation state (SDS) of such a plate caused by the action of an exterior load $X_n^{\pm}(t) + iY_n^{\pm}(t)$ $(t \in L)$ along the edges of the cuts and by specified stresses at infinity.

Let us assume that the edges of the cuts are not in contact* and the principal vector of the external stresses acting on an edge of the cuts is known. We shall also assume we are given the complex potentials $\Phi_{\nu 0}(z_\nu)$, giving a solution of the problem, for a continuous plate, of external stresses applied at infinity.

*In some problems it is necessary to impose a physical condition, excluding the possibility of an overlapping of the edges of a cut. Such problems are nonlinear and must be solved in an incremental setting, i.e., by stepwise changes of the loading on edges of the cuts.

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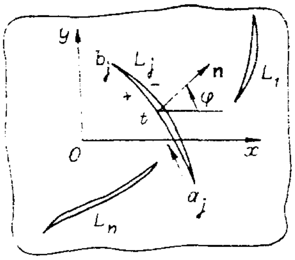

Fig. 1

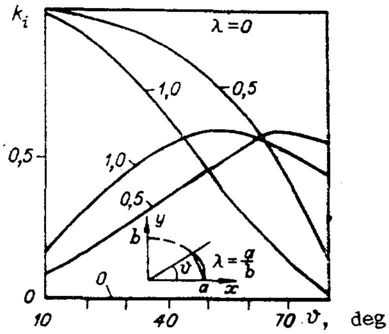

Fig. 2

To solve the stated problem for an arrangement of cracks L on curves, we specify certain continuously distributed dislocations and concentrated stresses and we determine the stress state caused by them in the region considered. We then select the Burgers components of the dislocations $\mathbf{Q}(t)$ and the concentrated stresses $\mathbf{P}(t)$ (or, equivalently, certain functions of them) so that the stresses on the edges of L, due to the dislocations and concentrated stresses, are equal to those specified.

At a point $z = \tau$ of an infinite anisotropic plate let there be applied a concentrated stress with vector $\mathbf{P}=X+iY$ or a concentrated dislocation with Burgers vector $\mathbf{Q}=U+iV$ [6]. We can write the Lekhnitskii potentials [7] stipulated by them, respectively, in the form

$$\Phi_\nu^*(z_\nu)=\frac{A_\nu}{z_\nu-\tau_\nu},\quad \Phi_\nu^{**}(z_\nu)=\frac{B_\nu}{z_\nu-\tau_\nu}, \tag{1.1}$$

where the complex constants A_ν, B_ν are determined from conditions of equilibrium and uniqueness of displacements:

$$\sum_{\nu=1}^{2}\{\mu_\nu^{k-2}(A_\nu, B_\nu)-\bar\mu_\nu^{k-2}(\bar A_\nu, \bar B_\nu)\}=(f_{1k}, f_{2k})(2\pi i)^{-1},$$
$$f_{11}=(a_{12}X+a_{26}Y)a_{22}^{-1},\quad f_{12}=Y,\quad f_{13}=X, \tag{1.2}$$
$$f_{14}=-(a_{16}X+a_{26}Y)a_{22}^{-1},\quad f_{21}=V,\quad f_{22}=f_{23}=0,\quad f_{24}=U.$$

From the latter relations it follows, in particular, that

$$A_0A_1-B_0\bar A_1+A_2=0,\quad a_0B_1-b_0\bar B_1+B_2=0,$$
$$A_0=(\bar p_2q_1-p_1\bar q_2)(\bar p_2q_2-p_2\bar q_2)^{-1},\quad B_0=(\bar p_2\bar q_1-\bar p_1\bar q_2(\bar p_2q_2-p_2\bar q_2)^{-1}, \tag{1.3}$$
$$a_0=(\mu_1-\bar\mu_2)(\mu_2-\bar\mu_2)^{-1},\quad b_0=(\bar\mu_1-\bar\mu_2)(\mu_2-\bar\mu_2)^{-1},$$
$$p_\nu=a_{11}\mu_\nu+a_{12}-a_{16}\mu_\nu,\quad q_\nu=a_{12}\mu_\nu+a_{22}\mu_\nu^{-1}-a_{26}\quad(\nu=1{,}2).$$

Here μ_ν are roots of the characteristic equation; a_{ij} are coefficients of deformation from Hooke's law [7].

Assuming that $\mathbf{P}(t)$, $\mathbf{Q}(t)$ are functions of class H on L [8], we find, with the aid of relation (1.2), the A(t), B(t) corresponding to them; we multiply relation (1.1) by ds and integrate along L. We obtain

$$\Phi_{\nu1}(z_\nu)=\int_L\frac{A_\nu(\tau)\,ds}{z_\nu-\tau_\nu},\qquad \Phi_{\nu2}(z_\nu)=\int_L\frac{B_\nu(\tau)\,ds}{z_\nu-\tau_\nu}. \tag{1.4}$$

Introducing the change of variables

$$\mu_\nu(t)=-2\pi iA_\nu(t)/M_\nu(t),\quad \omega_\nu(t)=-2\pi iB_\nu(t)/M_\nu(t),$$
$$d\tau_\nu=M_\nu(\tau)\,ds,\quad M_\nu(t)=\mu_\nu\cos\varphi(t)+\sin\varphi(t),$$

we rewrite relations (1.4), (1.3) in the form

$$\Phi_{\nu1}(z_\nu)=\frac{1}{2\pi i}\int_L\frac{\mu_\nu(\tau)\,d\tau_\nu}{\tau_\nu-z_\nu},\quad \Phi_{\nu2}(z_\nu)=\frac{1}{2\pi i}\int_L\frac{\omega_\nu(\tau)\,d\tau_\nu}{\tau_\nu-z_\nu}; \tag{1.5}$$

$$A(t)\mu_1(t)+B(t)\overline{\mu_1(t)}+\mu_2(t)=0; \tag{1.6}$$

$$a(t)\omega_1(t)+b(t)\overline{\omega_1(t)}+\omega_2(t)=0,\quad A(t)=A_0\frac{M_1(t)}{M_2(t)},\quad B(t)=B_0\frac{\overline{M_1(t)}}{M_2(t)},\quad a(t)=a_0\frac{\overline{M_1(t)}}{M_2(t)},\quad b(t)=b_0\frac{\overline{M_1(t)}}{M_2(t)}. \tag{1.7}$$

We may then represent the sought-for solutions of the problem formulated above as

$$\Phi_\nu(z_\nu)=\sum_{j=0}^{2}\Phi_{\nu j}(z_\nu). \tag{1.8}$$

Substituting the limit values of the functions $\Phi_\nu(z_\nu)$ from Eq. (1.8) into the boundary conditions on L

$$\begin{gathered} a(t)\Phi_1^{\pm}(t_1)+b(t)\overline{\Phi_1^{\pm}(t_1)}+\Phi_2^{\pm}(t_2)=F^{\pm}(t),\quad t\in L,\\ F^{\pm}(t)=\{X_n^{\pm}(t)+\bar{\mu}_2Y_n^{\pm}(t)\}\{(\mu_2-\bar{\mu}_2)M_2(t)\}^{-1}\end{gathered} \tag{1.9}$$

and subtracting the second equation from the first, we obtain

$$a(t)\mu_1(t)+b(t)\overline{\mu_1(t)}+\mu_2(t)=F_1(t),\qquad F_1(t)=F^{+}(t)-F^{-}(t). \tag{1.10}$$

Using the relations (1.6), (1.10), we readily establish that the functions $\mu_\nu(t)$ may be expressed explicitly in terms of jumps in the stresses $X(t) = X^+(t) - X^-(t)$, $Y(t) = Y^+(t) - Y^-(t)$ at the edges of the cuts.

Adding the two limit equations and subtracting the relation (1.7), we find, after some simplifications, the basic SIE of the problem:

$$\begin{gathered}
\int_L\frac{\omega_1(\tau)\,d\tau_1}{\tau_1-t_1}+\int_L\{K_{11}(t,\tau)\omega_1(\tau)+K_{12}(t,\tau)\overline{\omega_1(\tau)}\}\,ds=f_1^*(t),\\
K_{11}(t,\tau)\,ds=\frac{1}{2}\left\{d\ln\frac{\bar{\tau}_2-\bar{t}_2}{\tau_1-t_1}+\frac{b(\tau)-b(t)}{\overline{b(t)}(\bar{\tau}_2-\bar{t}_2)}d\bar{\tau}_2\right\},\\
K_{12}(t,\tau)\,ds=\frac{1}{2}\left\{d\ln\frac{\bar{\tau}_2-\bar{t}_2}{\bar{\tau}_1-\bar{t}_1}+\frac{\overline{a(\tau)}-\overline{a(t)}}{\overline{b(t)}(\bar{\tau}_2-\bar{t}_2)}d\bar{\tau}_2\right\},\\
f^*(t)=f_1(t)-\int_L\{K_{13}(t,\tau)\mu_1(\tau)+K_{14}(t,\tau)\overline{\mu_1(\tau)}\}\,ds,\\
K_{13}(t,\tau)\,ds=\frac{1}{2}\left\{\frac{d\tau_1}{\tau_1-z_1}+\frac{\overline{B(\tau)}\,d\bar{\tau}_2}{\overline{b(t)}(\bar{\tau}_2-\bar{t}_2)}\right\},\\
K_{14}(t,\tau)\,ds=\frac{1}{2}\left\{\frac{\overline{A(\tau)}\,d\bar{\tau}_2}{\overline{b(t)}(\bar{\tau}_2-\bar{t}_2)}-\frac{\overline{a(t)}\,d\bar{\tau}_1}{\overline{b(t)}(\bar{\tau}_1-\bar{t}_1)}\right\},\\
f_1(t)=\frac{\pi i}{2\overline{b(t)}}\{\overline{F_2(t)}-2[\overline{a(t)}\,\overline{\Phi_{10}(t_1)}+\overline{b(t)}\Phi_{10}(t_1)+\overline{\Phi_{20}(t_2)}]\},\\
F_2(t)=F^{+}(t)-F^{-}(t).
\end{gathered} \tag{1.11}$$

Taking relations (1.5) into account and the relations (see [7])

$$(u,v)=2\,\mathrm{Re}\left\{\sum_{\nu=1}^{2}(p_\nu,q_\nu)\varphi_\nu(z_\nu)\right\},\quad \Phi_\nu(z_\nu)=\varphi_\nu'(z_\nu)$$

we find that the jump in the displacements $w(t) = [(u + iv)^+ - (u + iv)^-]$ on L_j $(j = \overline{1, n})$ has the form

$$w(t)=\sum_{\nu=1}^{2}\left\{(p_\nu+iq_\nu)\int_{a_j}^{t}\omega_\nu(\tau)\,d\tau_\nu+(\bar{p}_\nu+i\bar{q}_\nu)\int_{a_j}^{t}\overline{\omega_\nu(\tau)}\,d\bar{\tau}_\nu\right\}. \tag{1.12}$$

Using the relations (1.12), we readily ascertain the physical essence of the unknown functions $\omega_\nu(t)$. Differentiating relation (1.12), we find that

$$\frac{dw}{ds}=\sum_{\nu=1}^{2}\{(p_\nu+iq_\nu)M_\nu(t)\omega_\nu(t)+(\bar{p}_\nu+i\bar{q}_\nu)\overline{M_\nu(t)}\,\overline{\omega_\nu(t)}\}. \tag{1.13}$$

It thus follows from relations (1.7), (1.13) that the values of $\omega_\nu(t)$ are directly connected with the derivatives of the jumps of the displacements on the edges of L.

From conditions of continuity of the displacements at the crack tips and relation (1.12) we obtain

$$\int_{L_j} \omega_1(\tau)\, d\tau_1 = 0 \quad (j = \overline{1, n}). \tag{1.14}$$

Equations (1.11) and (1.14) furnish a solution of the stated problem. For the particular case of a self-balanced continuous loading [$F_1(t) = \mu_\nu(t) = 0$] the results given in [1, 4], obtained by another method, follow from relations (1.5), (1.11), (1.14).

Since the number of conditions, which the unknown functions $\omega_1(t)$ must satisfy, coincides with the number of SIE over corresponding contours of L, the solution of system (1.11), (1.14) must then be sought in the class of functions unbounded at the ends of the cuts [8].

If the cuts are specified along a straight line, the solution of the problem, according to [8], can then be obtained in closed form. For example, for a plate with the crack $L = \{\alpha \leqslant x \leqslant \beta;\ y = 0\}$ along an interval of the real axis loaded by means of stresses $\sigma_y^{\pm}, \tau_{xy}^{\pm}, \sigma_y^{\infty}, \tau_{xy}^{\infty}$ over edges of the cut and at infinity, the integral equation (1.11) takes the form

$$\frac{1}{\pi i}\int_L \frac{\omega_1(\tau)\, d\tau}{\tau - t} = H(t) + \frac{1}{\pi i}\int_L \frac{h(\tau)\, d\tau}{\tau - t},$$

$$H(t) = \{(\tau_{xy}^{+} + \tau_{xy}^{-}) + \mu_2(\sigma_y^{+} + \sigma_y^{-}) + 2(\tau_{xy}^{\infty} + \mu_2\sigma_y^{\infty})\}\{2(\mu_1 - \mu_2)\}^{-1},$$

$$h(t) = \{(\tau_{xy}^{+} - \tau_{xy}^{-}) + \mu_2(\sigma_y^{+} - \sigma_y^{-})\}\{2(\mu_1 - \mu_2)\}^{-1},$$

and its general solution

$$\omega_1(t) = h(t) + \frac{1}{\pi i X(t)}\int_L \frac{X(\tau) H(\tau)}{\tau - t}\, d\tau + \frac{2C}{X(t)},$$

where C is an unknown constant.

After some transformations, we find

$$\Phi_1(z_1) = \frac{1}{2(\mu_1 - \mu_2)}\left\{-\frac{1}{2\pi i}\int_L \frac{\mu_2(\sigma_y^{+} - \sigma_y^{-}) + (\tau_{xy}^{+} - \tau_{xy}^{-})}{\tau - z_1}\, d\tau - \right.$$

$$\left. - \frac{1}{2\pi i X(z_1)}\int_L \frac{X(\tau)\left[\mu_2(\sigma_y^{+} - \sigma_y^{-}) + (\tau_{xy}^{+} - \tau_{xy}^{-})\right]}{\tau - z_1}\, d\tau + (\mu_2\sigma_y^{\infty} + \tau_{xy}^{\infty})\left[1 - \frac{2z_1 - (\alpha + \beta)}{2X(z_1)}\right] + \frac{C}{X(z_1)}\right\}, \tag{1.15}$$

$$\Phi_2(z_2) = \frac{1}{2(\mu_1 - \mu_2)}\left\{\frac{1}{2\pi i}\int_L \frac{\mu(\sigma_y^{+} - \sigma_y^{-}) + (\tau_{xy}^{+} - \tau_{xy}^{-})}{\tau - z_2}\, d\tau + \frac{1}{2\pi i X(z_2)}\int_L \frac{X(\tau)\left[\mu_1(\sigma_y^{+} + \sigma_y^{-}) + (\tau_{xy}^{+} + \tau_{xy}^{-})\right]}{\tau - z_2}\, d\tau - \right.$$

$$\left. - (\mu_1\sigma_y^{\infty} + \tau_{xy}^{\infty})\left[1 - \frac{2z_1 - (\alpha + \beta)}{2X(z_2)}\right] - \frac{(\mu_1 - \bar{\mu}_2)C - (\mu_1 - \mu_2)\bar{C}}{(\mu_2 - \bar{\mu}_2)X(z_2)}\right\}.$$

To determine C, we find, from conditions for uniqueness of displacements, the equation

$$C = \{(X + \bar{\mu}_2 Y)\left[\bar{A}(p_2 + iq_2 + B) + B(p_2 - iq_2 - \bar{A})\right] -$$

$$- (X + \mu_2 Y)\left[\bar{A}(p_2 + iq_2 - A) + B(\bar{p}_2 - i\bar{q}_2 + \bar{B})\right]\}\{2\pi i(|A|^2 - |B|^2)\}^{-1},$$

$$A = \{(p_1 + iq_1)(\mu_2 - \bar{\mu}_2) - (p_2 + iq_2)(\mu_1 - \bar{\mu}_2) +$$

$$+ (\bar{p}_2 + i\bar{q}_2)(\mu_1 - \mu_2)\}[2(\mu_1 - \mu_2)]^{-1}, \tag{1.16}$$

$$B = \{(\bar{p}_1 + i\bar{q}_1)(\mu_2 - \bar{\mu}_2) - (p_2 + iq_2)(\bar{\mu}_1 - \bar{\mu}_2) + (\bar{p}_2 + i\bar{q}_2)(\bar{\mu}_1 - \mu_2)\}[2(\bar{\mu}_1 - \bar{\mu}_2)]^{-1},$$

$$X + iY = \int_L (\tau_{xy}^{-} - \tau_{xy}^{+})\, d\tau + i\int_L (\sigma_y^{-} - \sigma_y^{+})\, d\tau.$$

Restricting the discussion to the case of an orthotropic material with $\mu_\nu = i\beta_\nu$ ($\nu = 1, 2$) and a crack $L = \{|x| < a;\ y = 0\}$, loaded along the upper edge of the interval $b < x < c$ ($-a < b < c < a$) by a constant pressure and by tangential stresses σ and τ, we obtain from relations (1.15), (1.16) the stress intensity coefficients (SIC) of separation and shear at the crack tips:

$$K_1(\pm a) = \frac{\sigma}{2}\sqrt{\frac{a}{\pi}}\left\{\pm\sqrt{1 - \left(\frac{b}{a}\right)^2} \mp \sqrt{1 - \left(\frac{c}{a}\right)^2} + \arcsin\frac{c}{a} - \right.$$

$$\left. - \arcsin\frac{b}{a}\right\} + \frac{\tau(c - b)}{2\pi\sqrt{a}}\left\{\frac{1}{(\beta_1 + \beta_2)}\left[\frac{a_{12}}{a_{22}}\beta_1\beta_2 + 1\right]\right\},$$

$$K_2(\pm a) = -\frac{\sigma(c-b)}{2\pi\sqrt{a}}\left\{\frac{1}{(\beta_1+\beta_2)}\left[\frac{a_{12}}{a_{11}}+\beta_1\beta_2\right]\right\}+$$
$$+\frac{\tau\sqrt{a}}{2\pi}\left\{\pm\sqrt{1-\left(\frac{b}{a}\right)^2}\mp\sqrt{1-\left(\frac{c}{a}\right)^2}+\arcsin\frac{c}{a}-\arcsin\frac{b}{a}\right\}.$$

2. We now consider the problem formulated above when on the cuts L there are specified the displacements

$$(u+iv)^{\pm} = G^{\pm}(t) = g_1^{\pm}(t) + ig_2^{\pm}(t),\ t\in L. \tag{2.1}$$

We shall assume that the principal vector of external stresses, acting on edges of the cuts L_j, is known: $X_j + iY_j$, and the functions $G^{\pm}(t)$ satisfy, at the tips a_j, b_j of cut L_j, the continuity condition

$$G^+(a_j) = G^-(a_j),\quad G^+(b_j) = G^-(b_j). \tag{2.2}$$

To the boundary conditions (2.1) we can attach the form

$$W^{\pm}(t) = \left(\bar{p}_2\frac{dg_2^{\pm}}{ds} - \bar{q}_2\frac{dg_1^{\pm}}{ds}\right)\{(\bar{p}_2 q_2 - p_2\bar{q}_2)M_2(t)\}^{-1}. \tag{2.3}$$

Substituting $\Phi_\nu(z_\nu)$ from Eq. (1.8) into Eq. (2.3), and subtracting the second equation from the first, we obtain

$$A(t)\omega_1(t) + B(t)\overline{\omega_1(t)} + \omega_2(t) = W_1(t),\quad W_1(t) = W^+(t) - W^-(t). \tag{2.4}$$

From relations (1.7), (2.4) we determine $\omega_\nu(t)$ and, proceeding, we assume that they and the potentials $\Phi_{\nu 2}(z_\nu)$ from relations (1.5) are known.

Adding the limit equations (2.3) and taking relation (1.6) into account, we find, similarly, the basic SIE of the problem:

$$\begin{gathered}
\int_L \frac{\mu_1(\tau)\,d\tau_1}{\tau_1 - t_1} + \int_L \{K_{21}(t,\tau)\mu_1(\tau) + K_{22}(t,\tau)\overline{\mu_1(\tau)}\}\,ds = f_2^*(t),\\
K_{21}(t,\tau)\,ds = \frac{1}{2}\left\{d\ln\frac{\bar{\tau}_2-\bar{t}_2}{\tau_1-t_1} + \frac{\overline{B(\tau)}-\overline{B(t)}}{\overline{B(t)}(\bar{\tau}_2-\bar{t}_2)}\,d\bar{\tau}_2\right\},\\
K_{22}(t,\tau)\,ds = \frac{1}{2}\left\{d\ln\frac{\bar{\tau}_2-\bar{t}_2}{\tau_1-t_1} + \frac{\overline{A(\tau)}-\overline{A(t)}}{\overline{B(t)}(\bar{\tau}_2-\bar{t}_2)}\,d\bar{\tau}_2\right\},\\
f^*(t) = f_2(t) - \int_L \{K_{23}(t,\tau)\omega_1(\tau) + K_{24}(t,\tau)\overline{\omega_1(\tau)}\}\,ds,\\
f_2(t) = \frac{\pi i}{2\overline{B(t)}}\{W_2(t) - 2[\overline{A(t)\Phi_{10}(t_1)} + \overline{B(t)}\Phi_{10}(t_1) + \overline{\Phi_{20}(t_2)}]\},\\
K_{23}(t,\tau)\,ds = \frac{1}{2}\left\{\frac{d\tau_1}{\tau_1-t_1} + \frac{b(\tau)\,d\bar{\tau}_2}{\overline{B(t)}(\bar{\tau}_2-\bar{t}_2)}\right\},\\
K_{24}(t,\tau)\,ds = \frac{1}{2}\left\{\frac{a(\tau)}{\overline{B(t)}}\frac{d\bar{\tau}_2}{\tau_2-t_2} - \frac{\overline{A(t)}}{\overline{B(t)}}\frac{d\bar{\tau}_1}{\bar{\tau}_1-\bar{t}_1}\right\},\\
W_2(t) = W^+(t) - W^-(t).
\end{gathered} \tag{2.5}$$

From relations (1.9), (2.2) it follows that the sought-for function must be subjected to the additional condition

$$\begin{gathered}
\int_L \mu_1(\tau)\,d\tau_1 = \Lambda_j\quad (j=\overline{1,n}),\\
\Lambda_j = \frac{(X_j+\mu_2 Y_j)[(\mu_1-\bar{\mu}_2)-(\mu_2-\bar{\mu}_2)A_0] + (X_j-\mu_2 Y_j)[(\mu_1-\bar{\mu}_2)-(\mu_2-\bar{\mu}_2)B_0]}{|(\mu_1-\mu_2)-(\mu_2-\bar{\mu}_2)A_0|^2 + |(\bar{\mu}_1-\bar{\mu}_2)-(\mu_2-\bar{\mu}_2)B_0|^2}.
\end{gathered} \tag{2.6}$$

Assume that on the cuts L_j ($j = \overline{1,k}$; $k < n$) the boundary conditions (1.9) are specified, and that on the cuts $L_{k+1},\ldots,L_n$ the conditions (2.3) are specified. In this case the sought-for solutions have the form (1.8). For determining the unknowns $\omega_\nu(t)$ for $t\in L_1 = \bigcup_{j=1}^{k} L_j$ and $\mu_\nu(t)$ for $t\in L_2 = \bigcup_{j=k+1}^{n} L_j$ we obtain the system of SIE

$$\int_{L_1} \frac{\omega_1(\tau)\,d\tau_1}{\tau_1 - t_1} + \int_{L_1} \{K_{11}(t,\tau)\omega_1(\tau) + K_{12}(t,\tau)\overline{\omega_1(\tau)}\}\,ds + \int_{L_2} \{K_{13}(t,\tau)\mu_1(\tau) + K_{14}(t,\tau)\overline{\mu_1(\tau)}\}\,ds = f_1^{**}(t_1),\ t \in L_1; \tag{2.7}$$

$$\int_{L_2} \frac{\mu_1(\tau)\,d\tau_1}{\tau_1 - t_1} + \int_{L_2} \{K_{21}(t,\tau)\mu_1(\tau) + K_{22}(t,\tau)\overline{\mu_1(\tau)}\}\,ds + \int_{L} \{K_{23}(t,\tau)\omega_1(\tau) + K_{24}(t,\tau)\overline{\omega_1(\tau)}\}\,ds = f_2^{**}(t),\ t \in L_2, \tag{2.8}$$

$$f_1^{**}(t) = f_1(t) - \int_{L_2} \{K_{13}(t,\tau)\mu_1(\tau) + K_{14}(t,\tau)\overline{\mu_1(\tau)}\}\,ds - \int_{L_2} \left\{\frac{\omega_1(\tau)\,\dot{\tau}_1}{\tau_1 - t_1} + K_{11}(t,\tau)\omega_1(\tau) + K_{12}(t,\tau)\overline{\omega_1(\tau)}\right\} ds,$$

$$f_2^{**}(t) = f_2(t) - \int_{L_2} \{K_{23}(t,\tau)\omega_1(\tau) + K_{24}(t,\tau)\overline{\omega_1(\tau)}\}\,ds - \int_{L} \left\{\frac{\mu_1(\tau)\,\dot{\tau}_1}{\tau_1 - t_1} + K_{21}(t,\tau)\mu_1(\tau) + K_{22}(t,\tau)\overline{\mu_1(\tau)}\right\} ds,\quad \dot{\tau}_\nu = \frac{d\tau_\nu}{ds}.$$

It is necessary to supplement Eqs. (2.7), (2.8) with the conditions (1.14) for $j = \overline{1, k}$ and with condition (2.6) for $j = \overline{k + 1, n}$.

We may solve, in a similar way, with the aid of the potentials (1.8), the problem arising when displacements are specified on one edge of the cuts of L and stresses are specified on the other edge.

The integral representations (1.5) constitute a general solution, and with their aid we can study very diverse boundary value problems for domains with cuts and linear inclusions. In particular, upon satisfying, with the aid of the representations (1.5) and formulas (1.9) and (2.3), the boundary conditions for an anisotropic plate with cuts, in which on one edge of a cut stresses are specified while on the other edge displacements are specified, we obtain SIE on the second kind. Upon using the conditions for non-ideal contact of elastic bodies, in which stresses and displacements of the edges of the cut are connected by linear relationships (see [3, 4, 9]), we can obtain singular integrodifferential equations of Prandtl type for bodies with thin-walled elastic inclusions. The integral representations obtained can also be applied in the solution of various mixed (contact) problems for bodies with cuts, problems involving strips of plasticity modeled by means of jumps in displacements [9].

3. The SIE (1.11), (2.5), (2.7), and (2.8) belong to the type of equations studied in detail in the literature [8]. In the class of functions unbounded close to the endpoints a_j, b_j (in the class H_0^*) the index of the SIE is equal to $\varkappa = 1$ (see [8]); these equations are always solvable and their solution on each arc L_j contains a single arbitrary constant, appearing linearly. With fulfillment of conditions (1.14), (2.6) the solution of the given equations is unique.

Thus the solution, for example, of Eqs. (1.11), (1.14) can be represented in the form

$$\omega_1(t) = \frac{\Omega_j^*(t)}{\sqrt{(t - a_j)(t - b_j)}},\quad t \in L_j \quad (j = \overline{1, n}),$$

where $\Omega_j^*(t)$ is a function of class H on L_j in a neighborhood of end-points of a cut; $\sqrt{(t - a)(t - b_j)}$ is an arbitrarily defined branch, varying continuously on L_j.

Let the equations of arcs L_j be described by the relations $t = \tau^j(\eta)$, $a_j = \tau^j(-1)$, $b_j = \tau^j(1)$ (η is a dimensionless real parameter). We assume functions $\tau^j(\eta)$ to be continuously differentiable on [−1, 1]. After a change of variables, Eq. (1.11) can be written in the form of a system of SIE:

$$\int_{-1}^{1} \frac{F_j(\xi,\eta)\chi_j(\eta)\,d\eta}{\eta - \xi} + \sum_{p=1}^{n} \int_{-1}^{1} \{k_1^{jp}(\xi,\eta)\chi_p(\eta) + k_2^{jp}(\xi,\eta)\overline{\chi_p(\eta)}\}\,d\eta = f_j(\xi),$$
$$\omega_1(t) = \omega_1[\tau^j(\xi)] = \chi_j(\xi) = \chi_j^0(\xi)(1 - \xi^2)^{-1/2}, \tag{3.1}$$

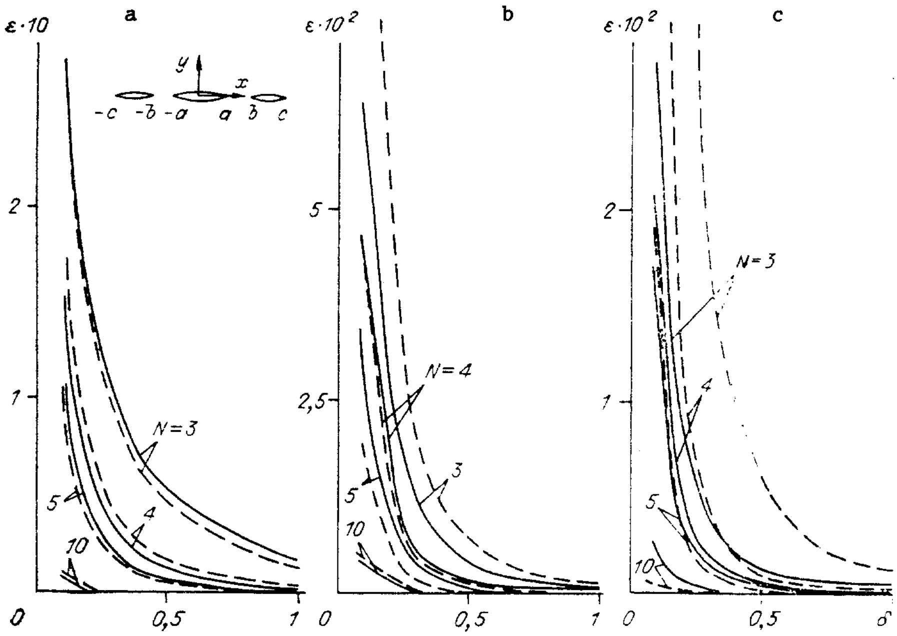

Fig. 3

$$F_j(\xi,\eta)=\frac{(\eta-\xi)\,\dot{\tau}_1^j(\eta)}{\tau_1^j(\eta)-\tau_1^j(\xi)},\quad \dot{\tau}=\frac{d\tau}{d\xi},\quad f_j(\xi)=f_j\left[\tau^j(\xi)\right],$$

$$k_1^{jp}(\xi,\eta)=\frac{ds}{d\eta}K_{11}\left[\tau^j(\xi),\tau^p(\eta)\right]+\frac{(1-\delta_{js})\,\dot{\tau}_1^p(\eta)}{\tau_1^p(\eta)-\tau_1^j(\xi)},$$

$$k_2^{jp}(\xi,\eta)=\frac{ds}{d\eta}K_{13}\left[\tau^j(\xi),\tau^p(\eta)\right]$$

(δ_p is the Kronecker symbol). We write conditions (1.14) as

$$\int_{-1}^{1}\chi_j(\eta)\,\dot{\tau}_1^j(\eta)\,d\eta=0\quad(j=\overline{1,n}). \tag{3.2}$$

With the aid of the Gauss–Chebyshev formulas we reduce the solution of Eqs. (3.1), (3.2) to the solution of a system of linear algebraic equations with respect to approximate values of the sought-for functions $\chi_j^0(\beta)$ ($j=\overline{1,n}$) at the Chebyshev nodes $\beta_i=\cos(2i-1)\pi/(2M))$ ($i=\overline{1,N}$). Theoretical estimates of the convergence of this numerical method are given, for example, in [10].

Knowing $\chi_j^0(\beta_i)$ and $\chi_j^0(\pm 1)$, we obtain, based on relations (1.5), (1.6), and (2.4), asymptotic formulas for stresses in the neighborhoods of $c=\tau^j(\pm 1)$, the endpoints of the cut L_j (in what follows we omit the subscript j):

$$\lim\sqrt{2r}\,(\sigma_x,\tau_{xy},\sigma_y)=\operatorname{Re}\left\{\left(\pm\frac{ds}{d\eta}\Big|_{\eta=\pm1}\right)^{1/2}\sum_{\nu=1}^{2}\left(\mu_\nu^2,-\mu_\nu,1\right)C_\nu(\vartheta)\right\},$$

$$C_\nu(\vartheta)=\Omega_\nu\left[M_\nu(c)\,(\cos\vartheta+\mu_\nu\sin\vartheta)^{-1}\right]^{1/2},\ z-c=re^{i\vartheta},$$

$$\Omega_1=\chi^0(\mp1)+\psi^0(\mp1),\quad \psi^0(\eta)=\mu_1\left[\tau(\eta)\right](1-\eta^2)^{1/2},$$

$$\Omega_2=-a(c)\chi^0(\mp1)-b(c)\overline{\chi^0(\mp1)}-A(c)\psi^0(\mp1)-B(c)\overline{\psi^0(\mp1)},$$

we also obtain values of the SIC of separation and shear, namely, K_1, K_2 [4].

The SIE, obtained above, for basic problems of elasticity theory for systems of smooth curvilinear cuts can be also used for handling piecewise-smooth curvilinear cuts (broken-line cuts and branching cuts). Here a cut is split up into smooth portions having common points of intersection; also, application is made of a well-recommended simplified procedure for numerically solving the SIE (3.1) and (3.2) that arise, with fixed singularities [9], making it possible to bypass a study of the nature of the behavior of the sought-for functions in neighborhoods of corner points.

TABLE 1

Δ	δ	N: 3	4	5	3	4	5
		ε(±a)			ε(±b)		
1	0,1 0,5 1,0	0,27 0,05 0,02	0,15 0,01 0	0,10 0 0	0,27 0,05 0,01	0,17 0,02 0	0,10 0 0
0,1	0,1 0,5 1,0	0,06 0 0	0,04 0 0	0,03 0 0	0,12 0,01 0	0,05 0 0	0,02 0 0
0,05	0,1 0,5 1,0	0,03 0 0	0,02 0 0	0,01 0 0	0,11 0 0	0,05 0 0	0,02 0 0

The representations (1.5) and (1.8), and the algorithms for numerically solving the pertinent SIE, have proved to be an effective instrument for determining the SDS in a neighborhood of endpoints of cuts of complex form in anisotropic and isotropic plates. By way of illustration, we present below the results of a numerical solution of a series of model problems of fracture mechanics.

In Fig. 2, for plates of an orthotropic material ($E_x = E_1$ = 53.84 GPa, $E_y = E_2$ = 17.95 GPa, G_{12} = 8.63 GPa, $\nu_{yx} = \nu_1$ = 0.25), we present correction functions for the SIC for separation and shear, $k_{1,2} = K_{1,2}/(\sigma\sqrt{\pi\ell^*})$ (descending and ascending curves, $2\ell^*$ is the crack length) at the upper endpoint of a crack L, situated on an arc of an ellipse [a_1 = a, b_1 = $(a\cos\vartheta + ib\sin\vartheta)$, crack tip], as functions of a parameter ϑ for various ratios of ellipse semi-axes, $\lambda = a/b$, for the loading $\sigma_x^\infty = \sigma$, $\sigma_y^\infty = \tau_{xy}^\infty = 0$. The horizontal curves $k_1 = 1$, $k_2 = 0$ correspond to the value $K_{1,2}$ at the ends of a rectilinear crack ($\lambda = 0$). The calculations testify to the good convergence of the algorithm for weakly and strongly anisotropic materials ($1 \leqslant E_1/E_2 \leqslant 25$): already for $N \geqslant 5$ we have agreement in the first three significant figures.

We consider a symmetric system of three cracks $L_1 = \{|x| < a;\ y = 0\}$, $L_{2,3} = \{b < |x| < c;\ y = 0\}$, loaded by a constant pressure (Fig. 3), and we limit the discussion to the case of an orthotropic material with $\mu_\nu = i\beta_\nu$ ($\beta_\nu > 0$) for $(c - b) < 2a$ (the end cracks are not larger than the central crack). From relations (1.8), (1.11), and (1.14) we find that ($\lambda = 3 - \nu$)

$$\Phi_\nu(z_\nu) = \frac{\mu_\lambda\sigma}{\mu_\nu - \mu_\lambda}\,\frac{z_\nu^3 + Cz_\nu - R(z_\nu)}{2R(z_\nu)},$$

$$C = -\int_b^c \frac{x^3\,dx}{\sqrt{(x^2-a^2)(x^2-b^2)(c^2-x^2)}}\left\{\int_a^b \frac{x\,dx}{\sqrt{(x^2-a^2)(x^2-b^2)(c^2-x^2)}}\right\}^{-1},$$

$$R(z) = \sqrt{(z^2-a^2)(z^2-b^2)(z^2-c^2)},\quad R(z)\to z^3,\quad z\to\infty.$$

The integrals in C can be put in the form of complete elliptic integrals. The stress distribution σ_y on the x axis coincides with the distribution for an isotropic body, and the SIC for separation at the tips x = ±a, y = ±b of the cracks L_1, $L_{2,3}$ assume the form

$$K_1^0(\pm a) = \sigma\sqrt{\pi a}\,(c^2-a^2)^{1/2}(b^2-c^2)^{-1/2}E(k)/K^*(k),$$

$$K_1^0(\pm b) = \sigma\sqrt{\pi b}\,(b^2-a^2)^{1/2}(c^2-b^2)^{-1/2}\{(c^2-a^2)E(k)[(b^2-a^2)K^*(k)]^{-1}-1\},$$

where $K^*(k)$ and $E(k)$ are complete elliptic integrals of the first and second kinds with modulus $k = (c^2 - b^2)^{1/2}(c^2 - a^2)^{-1/2}$ [11].

A comparison of the calculated values of $K_1(\pm a)$, $K_1(\pm b)$ with the exact values of K_1^0 is shown in Fig. 3 and in Table 1. As a measure of the error of the numerical solution we take the relative error $\varepsilon = |K_1 - K_1^0|(K_1^0)^{-1}$ at the crack tips a and b (continuous and dashed curves) as a function of the relative size of the cross section $\delta = (b - a)/a$ for a distinct number N of subdivision nodes and crack-size ratios $\Delta = (c - b)(2a)^{-1}$ = 1; 0.1; 0.5 (Fig. 3a, 3b, and 3c, respectively). The calculations show that for a cross-connection between tips of the cracks L_1 and $L_{2,3}$, ($\delta > 0.2$), that is not too small and with $\Delta < 1$, satisfactory accuracy is already attained when $N \geqslant 3$. As Δ decreases and δ increases convergence gets better.

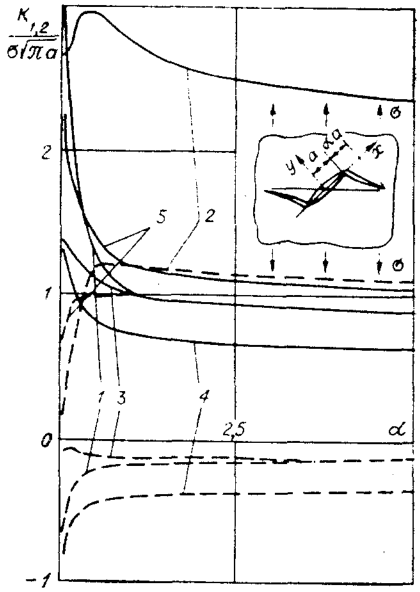

Fig. 4

In Fig. 4 we present the results of calculations for the SIC of separation and shear, $K_{1,2}$ (continuous and dashed curves) at the tips of a three-link broken-line crack, consisting of two curvilinear cuts (along arcs of a hyperbola with identical axes of magnitude a) starting from the endpoints of rectilinear cut $\Lambda = \{z = a\eta e^{i(\pi/4)} \mid |\eta| < 1\}$ as a function of parameter α, characterizing the size of the defect. The data are shown for an isotropic ($E_1/E_2 \to 1$, curve 1) and for an orthotropic material (boroplastic: E_1 = 276.1 GPa, E_2 = 27.61 GPa, G_{12} = 10.35 GPa, ν_1 = 0.25) for angle values $\varphi = 0$; $\pi/4$; $\pi/2$; $3\pi/4$ (curves 2-5) formed by the principal direction of anisotropy E_1 with the x axis. Calculations for strong (E_1/E_2 = 10) and weak ($E_1/E_2 \to 1$) anisotropic material show that already for $N \geqslant 10$ (N is the number of Chebyshev nodes at smooth links) coincidence of K_1 to the first three significant figures is observed.

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